Convergence Criteria for Iterative Processes

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Introduction

DURING the last few years, a considerable amount of effort has been put into the development of new methods of analysis for nonlinear structural systems. Taking advantage of the capabilities of modern electronic computers, these methods have been widely applied to problems of large deformations and stability, elasto-plasticity, creep, etc. Often, the numerical solution of these nonlinear problems is based on some iterative process or it involves the combination of an incremental and an iterative procedure.

An ever-recurrent problem associated with iterative techniques is the decision as to whether the current iterate is sufficiently close to the root without knowing the true solution itself. In the following, a discussion of the convergence problem will be given and some practical convergence criteria will be presented. For simplicity, a pure displacement formulation of the structural problem will be assumed.

Basic Methods

The convergence criteria used for nonlinear structural problems that are solved by iteration can usually be classified in one of the following three groups: 1) force criteria; 2) displacement criteria and 3) stress criteria.

The force criteria are normally based on a comparison between the current unbalanced or residual forces within the structure and the external loads. Making use of such a comparison does not always make sense because the force quantities to be compared may be of completely different order or even of different dimensions. For instance, it would be difficult to decide automatically whether the moments and out-of-plane forces within a buckled plate had converged with sufficient accuracy if the only external loading were lying in the plane of the plate. It would seem more consistent to compare the residual forces with the stiffness properties of the structure. However, this in fact corresponds to working with displacements so the direct use of a displacement criterion would appear to be preferable. Some special displacemet criteria will be described in the next section.

The third criterion involves a check on changes in stress values during an iteration cycle; these changes can be compared with prescribed stress levels. This type of criterion is well suited for truss, cable and membrane structures during very large deformations.

Displacement Criteria

Consider a nondimensional vector based on displacements, defined as follows[1]:

$$\varepsilon^{(j)} = \left\{ \frac{\Delta r_1}{r_{1,\text{ref}}}, \frac{\Delta r_2}{r_{2,\text{ref}}}, \ldots \frac{\Delta r_i}{r_{i,\text{ref}}}, \ldots \frac{\Delta r_N}{r_{N,\text{ref}}} \right\}_{(j)}^{T} \tag{1}$$

where N is the total number of unknown components, and $\Delta r_{i(j)}$ is the change in displacement component i during iteration cycle j. Every such component is scaled by a reference displacement quantity $r_{i,\text{ref}}$. These reference quantities are, in general, not equal to the corresponding total components because if r_i is close to zero, the ratio $\Delta r_i / r_i$ could be a large number even after convergence has occurred. Instead, every Δr_i is scaled by the largest displacement component of the corresponding "type". For instance, in a plate problem involving deflections, rotations and inplane displacements, the deflection would be scaled by the largest deflection component, the rotations by the largest rotation, etc. A mean value could be used instead of a maximum value for such a scaling.

Three alternative norms are now suggested for measuring the size of the ε-vector.

Modified absolute norm:

$$\|\varepsilon\|_1 = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\Delta r_i}{r_{i,\text{ref}}} \right| \tag{2}$$

Modified Euclidean norm:

$$\|\varepsilon\|_2 = \left[\frac{1}{N} \sum_{i=1}^{N} \left| \frac{\Delta r_i}{r_{i,\text{ref}}} \right|^2 \right]^{1/2} \tag{3}$$

Maximum norm:

$$\|\varepsilon\|_\infty = \max_i \left| \frac{\Delta r_i}{r_{i,\text{ref}}} \right| \tag{4}$$

The absolute and the Euclidean norms have been modified by division by N to obtain quantities that are independent of the total number of components. For any of these norms, the following convergence criterion can be used.

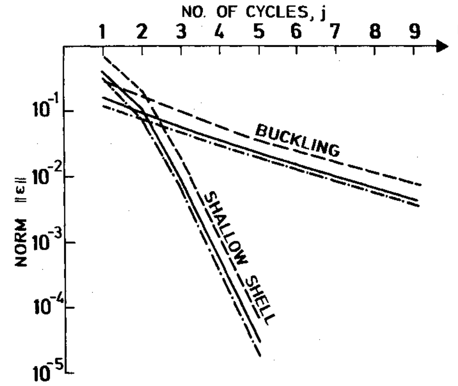
$$\|\varepsilon\| < \gamma \tag{5}$$

The value of γ will usually be of order 10^{-2} to 10^{-6}, depending on the desired accuracy.

Figure 1 demonstrates some typical uses of the norms defined by Eqs. (2, 3 and 4). The figure shows the values of these norms for one case of relatively low convergence (a plate buckling problem) and one case of fast convergence (a stable shallow shell problem). A Newton-Raphson iteration[2] was used. It is worth noting that the convergence rapidly stabilizes to a linear relation in this semilogarithmic plot. This behavior always was found when the iterate is sufficiently close to the root. The same type of behavior also was found to apply to modified Newton-Raphson schemes in which the gradient is kept constant during several iteration cycles.

For a true Newton-Raphson iteration it can be shown that the convergence is at least of second order when the iterate is sufficiently close to the (single) root and when the Jacobian (the gradient) satisfies certain conditions.[2]

It should also be noted from Fig. 1 that the various norms follow each other in a parallel manner during iteration. This indicates that it is of no great significance what specific norm is being used; one can account for the type of norm when choosing the magnitude of γ. The maximum norm might be



---- = MAXIMUM NORM
——— = MODIFIED EUCLIDEAN NORM
—·—·— = MODIFIED ABSOLUTE NORM

Fig. 1 Comparison of convergence criteria.

Received February 2, 1972.
Index categories: Structural Static Analysis; Structural Stability Analysis.

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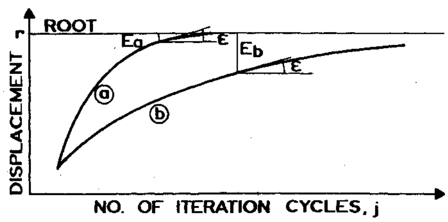

Fig. 2 Examples of slow and fast convergence.

slightly preferable since the other norms yield more of an average bound instead of an absolute bound for all displacement components.

Integrated Norm Criterion

The ε-vector as defined in Eq. (1) does in fact represent the change in displacements during one iteration cycle and it is not an expression of the distance between the present iterate and the true root. As shown in Fig. 2, two cases with the same $\|\varepsilon\|$ (represented by the slope of the curves) may have different total error E_a and E_b. However, an expression for the total error may be obtained taking the integral

$$E_{(j)} \leqq \int_{n=j}^{\infty} \|\varepsilon\| \, dn \tag{6}$$

where n is the iteration cycle.

Since it was demonstrated that the convergence is linear in a semilogarithmic plot, $\|\varepsilon\|$ can be approximated by

$$\|\varepsilon\| \simeq e^{\alpha-\beta n} \quad \text{or} \quad \log \|\varepsilon\| \simeq \alpha - \beta n \tag{7}$$

The coefficients α and β can be obtained by using two of the known values of the curve or by determining a "best fit" through previous values. Substituting Eq. (7) into Eq. (6) and carrying out this extrapolated integration yields

$$E_{(j)} \leqq \frac{1}{\beta} e^{\alpha-\beta j} \tag{8}$$

The use of such an integrated maximum norm for one case of fast convergence and one case of slow convergence is demonstrated in Fig. 3. Thus, at cycle j, $E_{(j)}$ represents the sum of $\|\varepsilon\|_\infty$ at all subsequent cycles. Note that in case of slow convergence $E_{(j)}$ is larger than $\|\varepsilon\|_\infty$ whereas the opposite is true for fast convergence. The integrated norm therefore gives a more true picture of the total error. It may result in a overestimation of the true error when the convergence is oscillatory. However, the convergence stabilizes to a monotonic scheme in almost all cases.

In problems of structural stability, the load-deflection curve rapidly changes character at various levels of loading. As a rule, the convergence is slow in regions of low tangential stiffness. In such regions it usually is not necessary to carry out the iteration to a complete convergence, the difference between the true and the computed load-deflection curve will be small anyway. Unless an absolute error bound is required, the convergence criteria based on Eqs. (2, 3 or 4) and Eq. (5) will in such cases work very well for practical purposes.

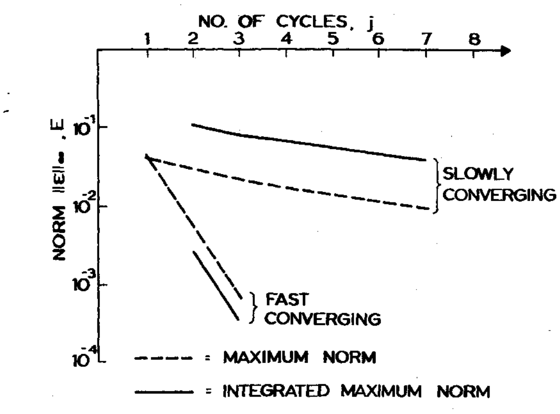

Fig. 3 Integrated error norm.

Conclusion

A discussion of several convergence criteria for the iterative solution of nonlinear structural problems has been given. In many cases, it is most efficient and accurate to base the convergence criterion merely on displacement quantities. Three alternative norms to be used in such a criterion have been suggested; all of them turn out to be highly useful for practical applications. A convergence criterion based on an extrapolated integration of these norms has also been proposed. This criterion should be used when an absolute bound on the error is desired.

References

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Heat Transfer at Reattachment of a Compressible Flow over a Backward Facing Step with a Suction Slot

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BASE-FLOW problems have been extensively studied in the last two decades by both analytical and experimental techniques. Yet the difficulty of such problems has forced the investigators to treat separately the regions of separation, reattachment, recirculating flow, free shear layer and outer flow, and match the individual solutions along the relevant interfaces. The analysis becomes even more complex if $\delta/h \sim 1$ because then the whole recirculating flow is well within the region of influence of viscosity and hence the inviscid core is eliminated. In this case the regions of separation and reattachment overlap and the expansion of the compressible flow is immediately followed by recompression. This is based on experimental evidence[1] that indicate a sudden drop of pressure at the corner followed immediately by an increase without any region of isobaric flow, which is commonly observed for $\delta/h \sim 0.1$ or smaller. The present Note suggests a simplified model for engineering estimates of the heat transfer in the neighborhood of reattachment for the case of supersonic flow over a backward facing step with a suction slot.

It is assumed that the flow properties in the vicinity of reattachment are similar to the properties of stagnation flow. It is also assumed that for $\delta/h \sim 1$ the heat exchanged along the stagnation streamline (SSL) is negligible and therefore the heat transfer at reattachment should be approximately equal to the heat transfer at the point of stagnation of a freestream, with properties those at the stagnation streamline before the expansion (see Fig. 1) which according to Ref. 3 and in the same notation is written as

$$q_R = (T_{os} - T_w)(-S_w'/S_w)(k/x)(Re_x)^{1/2} \{[(m+1)/2]\, d\ln X/d\ln x\}^{1/2} \tag{1}$$

Received Febraury 7, 1972. This work was partially supported by NASA Grant NGR-47-004-070. The contribution of J. A. Schetz is acknowledged for some initial basic ideas.

Index categories: Boundary Layers and Convecture Heat Transfer—Laminar; Sets, Waves, and Vicid-Inviscid Flow Interactions.

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